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Perpetual game options with a multiplied penalty

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ABSTRACT

The purpose of this paper is to examine a special kind of game option, whose main feature is the presence of an early exercise right for the seller as well as for the buyer. The seller has to pay some amount above the usual option payment for this right. Usually, this penalty payment is presented by a constant amount during the option life. Alternatively, in this paper we present the cancellation payment as the usual option payment multiplied by a constant. We introduce also a discount factor which gives a benefit for early exercising. It is closely related to the existence of a continuous dividend payment. In that way we can describe a dividend model in our framework.

The approach we use is based on finding the seller's and buyer's exercise regions. We do this maximizing their expected future financial results. After that we use the first exit properties to calculate the fair option price.

1. Introduction

The game options belong to the large class of derivatives which admit early exercising – all kinds of American derivatives, different convertible bonds and so on. There is a strong interest in this area in the recent years which is confirmed by the large amount of publications – see for example Yoon [22], Klimsiak et al. [13], Park and Jeon [17], Le and Dang [15], Balajewicz and Toivanen [1], Gong and Zhuang [9], Kang et al. [11], Zhao and Yang [24], Chen et al. [5], Madi et al. [16], Soleymani et al. [19], Chan [3], Chen et al. [4], and Gao et al. [8].

The main feature of the game options is the early exercise right for the seller in addition to the existing similar buyer's right. This right costs to the seller some amount above the usual option payment which is called a penalty. First Kifer [12] introduces these options as a particular case of the Dynkin stochastic games. Along with his seminal work, many authors examine this kind of options assuming that the penalty is a constant – we refer to Kallsen and Kühn [10], Kühn [14], [6], Suzuki and Sawaki [20], Emmerling [7], Yam et al. [21], and Zaevski [23]. We shall name these options ordinary game options.

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We assume in this paper that the maturity time is infinity. This means that both participants can exercise at every future moment. That allows them to wait sufficiently long time the asset price to hit the desired levels.

The main novelties of this paper are in two directions. The first one is adding an additional discount factor with the positive rate $\lambda > 0$ to the classical models of Kifer [12]. It has a significant role for financial modeling in several aspects. First, it provides a time dependence in the payment structure of the option. In such a way the possible future option payments vanish at infinity. This is related to the second importance of discounting. Namely, in the absence of maturity, it is the only factor which makes earlier exercising more attractive. The higher the discount rate is, the larger is the benefit of early exercising. Last but not least, the use of a discount factor allows us to examine a model with continuous dividend payments as a discounted non-dividend model. This is achieved by changing the model parameters. For this we permit negative values of the risk free rate of return.

The outstanding novelty of the paper is the form of the seller's cancellation penalty. We forsake the traditional assumption that the amount, which the seller owes for the early exercise right, is constant during the option life. Alternatively, we assume that this cancellation penalty is proportional to the amount which the buyer would receive if he/she decides to exercise. Thus we introduce a penalty dependance on the underlying asset price as well as the already introduced time dependence. In such a way the seller has to pay relatively small amount to cancel the contract if the underlying asset price is near the strike. This provides an additional right to him/her. Otherwise, the seller has to pay a larger penalty to cancel the contract if the buyer could achieve a relatively large profit deciding to exercise the contract immediately. This is an extra insurance for the buyer in addition to his/her early exercise right. Thus we significantly change the option payment structure and therefore the new game options are qualitatively different from the existed ones.

Our goal in the examination of such options is first to derive the regions in which the immediate exercise is optimal. For different values of the parameters the form of these regions can be quite different. For the call style options the seller's exercise region is $(0, A]$ for some constant A larger than the strike K, $K \le A < \infty$. Note that differently to ordinary game options, the values below the strike are always optimal for the seller. Another difference is that the seller's optimal boundary is finite even in the undiscounted case $\lambda = 0$. The buyer's exercise region is of the form (B, ∞), where B is another constant $A \le B$. This boundary is infinity when $\lambda = 0$ as well as when the options are ordinary. It turns out that both boundaries A and B coincide with the strike when the risk free rate is negative. This means that the points above the strike are optimal for the buyer, whereas the rest are optimal for the seller.

The optimal regions stay similar but inverse for the put style options. The seller's exercise region turns to $[B, \infty)$ for some constant B less than the strike, $B \le K$. The buyer's region is $(0, A)$, where $A \le B$. If the risk free rate is positive, then $A = B = K$. This means that the points below the strike are buyer's optimal, whereas the rest are seller's optimal.

Our approach to deriving the exercise regions is based on finding optimal stopping times. Let ζ be a stopping time. We shall denote by $A(\zeta)$ the seller's optimal strategy if the buyer has a strategy ζ. Similarly, we denote by $B(\zeta)$ the buyer's optimal strategy assuming that the seller has a strategy ζ. We search for a stopping time such that $\zeta = A(B(\zeta))$.

Note that we can restrict our examination to the first hitting times to some levels. This is due to (A) the Markovian nature of our asset processes, (B) the fact that the time dependance in the option payments is presented by a multiplication with $e^{-\lambda t}$, and (C) the infinite time horizon. In that way the optimal boundaries are constants. Also, these facts show that we can examine only the case $t = 0$.

Although we assume that the asset price is driven by a Brownian motion, some results are true for an arbitrary Markov process too. Also, some of the results are true for a finite time horizon too. We discuss briefly the case when the maturity is finite.

The paper is organized as follows. In Section 2 we present the base we shall use later. In Section 3 we derive the optimal regions. In Section 4 we derive the fair option price of a game call option, paying special attention of the undiscounted case in Section 5. The game put options are examined in Section 6. Some numerical results are presented in Section 7. Some propositions related to the first exit time of a Brownian motion are presented in Appendix A. In Appendix B we prove that the equations which determine the optimal boundaries have unique solutions.

2. Preliminaries

Let the asset price S_t be driven by a Feller-Markov process under the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$. Let the probability measure Q be risk neutral, the risk-free rate be a not necessarily positive constant r, and the discount rate be a positive constant $\lambda > 0$. We assume also that the total rate is positive, $r + \lambda > 0$. Let the penalty coefficient be the constant $\eta > 1$. Let $T < \infty$ be some finite moment. We can view it as a maturity date. Let the set $\mathcal{T}_{[t,T]}$ consists of all stopping times with values between t and T. The payments for discounted game call and put options are

$$\begin{aligned} N_1(t,x) &= e^{-\lambda t}(x-K)^+ \\ N_2(t,x) &= e^{-\lambda t}\eta(x-K)^+ \end{aligned} \tag{1}$$

and

$$\begin{aligned} N_1(t,x) &= e^{-\lambda t}(K-x)^+ \\ N_2(t,x) &= e^{-\lambda t}\eta(K-x)^+, \end{aligned} \tag{2}$$

respectively. We shall impose the following natural condition.

Condition 2.1. If an asset starts from a higher value, then its future value is higher too. Also if the initial asset value tends to zero/infinity, then the future asset value also tends to zero/infinity.

The next proposition gives the time dependence of the option prices.

Proposition 2.1. *Assume that the current asset price in moment t is $S_t = x$. Let the function $Y(t, x)$ present the price of a live perpetual game option. Then $Y(t,x) = e^{-\lambda t} Y(0,x)$.*

Proof. The proof can be found in Zaevski [23], proposition 2.2. □

Let us define the optimal strategies as follows.

Definition 2.1.

1. For every stopping time $\zeta \in \mathcal{T}_{[t,T]}$ we define the ζ-seller's optimal strategy $A(\zeta; x)$ as the stopping time which minimizes

$$E^{t,x}\begin{bmatrix} e^{-r(\zeta-t)} N_1\big(\zeta, S_\zeta\big) I_{\zeta \le A(\zeta;\cdot)} \\ + e^{-r(A(\zeta;\cdot)-t)} N_2\big(A(\zeta;\cdot), S_{A(\zeta;\cdot)}\big) I_{A(\zeta;\cdot)<\zeta} \end{bmatrix}. \tag{3}$$

2. Analogously, we define the ζ-buyer's optimal strategy, $B(\zeta; x)$, as the stopping time which maximizes

$$E^{t,x}\begin{bmatrix} e^{-r(B(\zeta;\cdot)-t)} N_1\big(B(\zeta;\cdot), S_{B(\zeta;\cdot)}\big) I_{B(\zeta;\cdot)\le\zeta} \\ + e^{-r(\zeta-t)} N_2\big(\zeta, S_\zeta\big) I_{\zeta<B(\zeta;\cdot)} \end{bmatrix}. \tag{4}$$

Remark 2.1. The uniqueness of the defined above stopping times is not important for our purposes. If one of them does not exist we shall assume that it is equal to the terminal date T or to infinity.

We denote by Υ^s and Υ^b the seller's and buyer's optimal regions and by $\overline{\Upsilon}$ the continuation region. We define them as follows.

Definition 2.2.

1. The point $(t, x) \in \Upsilon^b$ if for every stopping time $\zeta_1 \in \mathcal{T}_{[t,T]}$,

$$N_1(t,x) \ge E^{t,x}\begin{bmatrix} e^{-r(\zeta_1-t)} N_1\big(\zeta_1, S_{\zeta_1}\big) I_{\zeta_1 \le A(\zeta_1;\cdot)} \\ + e^{-r(A(\zeta_1;\cdot)-t)} N_2\big(A(\zeta_1;\cdot), S_{A(\zeta_1;\cdot)}\big) I_{A(\zeta_1;\cdot)<\zeta_1} \end{bmatrix}. \tag{5}$$

2. The point $(t, x) \in \Upsilon^s$ if for every stopping time $\zeta_2 \in \mathcal{T}_{[t,T]}$,

$$N_2(t,x) \le E^{t,x}\begin{bmatrix} e^{-r(B(\zeta_2;\cdot)-t)} N_1\big(B(\zeta_2;\cdot), S_{B(\zeta_2;\cdot)}\big) I_{B(\zeta_2;\cdot)\le\zeta_2} \\ + e^{-r(\zeta_2-t)} N_2\big(\zeta_2, S_{\zeta_2}\big) I_{\zeta_2<B(\zeta_2;\cdot)} \end{bmatrix}. \tag{6}$$

3. We define the continuation region as $\overline{\Upsilon} = \{[0,T] \times \mathbb{R}^+\}/\left\{\Upsilon^b \cup \Upsilon^s\right\}$.

Note that all defined above sets are disjoint.

3. Exercise regions

From now on we shall assume that $t = 0$. We can do this without loss of generality due to the Markovian property of the underlying process. For simplicity, we shall omit the time dependence in the pairs of the state space.

Now we shall prove some propositions related to the optimal regions.

3.1. Call style options

Suppose first that $\lambda = 0$. Proposition 3.1 from Zaevski [23] shows that the buyer's exercise region is empty.

Proposition 3.1. *If the discount factor is zero, $\lambda = 0$, then the buyer's exercise region is empty – $\Upsilon^b = \emptyset$.*

Note that all points less than the strike are optimal for the seller, because he has to pay nothing.

Proposition 3.2. *If $x \le K$, then $x \in \Upsilon^s$.*

The following proposition provides an important feature of the seller's exercise region.

Proposition 3.3. *Suppose that $x \in \Upsilon^s$ and $0 < y < x$. Then $y \in \Upsilon^s$ too.*

Proof. Note that Proposition 3.2 allows us to examine only the case $K < y < x$. Let T be a fixed finite moment, $\zeta \in \mathcal{T}[0,T]$ be arbitrary, and $B(\zeta; x)$ be the ζ-buyer's optimal strategy. Thus condition (6) leads to

$$E^x\begin{bmatrix} e^{-(r+\lambda)B(\zeta;x)}\big(S_{B(\zeta;x)} - K\big)^+ I_{B(\zeta;x)\le\zeta} \\ + e^{-(r+\lambda)\zeta}\eta\big(S_\zeta - K\big)^+ I_{\zeta<B(\zeta;x)} \end{bmatrix} \ge \eta(x-K).$$

Note that since the discounted asset price is a martingale and the stopping time $\zeta \wedge B(\zeta; y)$ is finite, we have

$$E^y\left[e^{-rB(\zeta;y)}S_{B(\zeta;y)}I_{B(\zeta;y)\le\zeta} + e^{-r\zeta}S_\zeta I_{\zeta<B(\zeta;y)}\right] = y. \tag{7}$$

Using Eq. (7), Condition 2.1, and the fact that $B(\zeta; y)$ maximizes Eq. (4) we derive

$$\begin{aligned}
&E^y\begin{bmatrix} e^{-(r+\lambda)B(\zeta;y)}\left(S_{B(\zeta;y)}-K\right)^+ I_{B(\zeta;y)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\eta\left(S_\zeta - K\right)^+ I_{\zeta<B(\zeta;y)}\end{bmatrix} - \eta(y-K) \\
&= E^y\begin{bmatrix} e^{-(r+\lambda)B(\zeta;y)}\left(S_{B(\zeta;y)}-K\right)^+ I_{B(\zeta;y)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\eta\left(S_\zeta - K\right)^+ I_{\zeta<B(\zeta;y)}\end{bmatrix} \\
&\quad - \eta E^y\left[e^{-(r+\lambda)B(\zeta;y)}e^{\lambda B(\zeta;y)}S_{B(\zeta;y)}I_{B(\zeta;y)\le\zeta} + e^{-(r+\lambda)\zeta}e^{\lambda\zeta}S_\zeta I_{\zeta<B(\zeta;y)}\right] + K\eta \\
&= E^y\begin{bmatrix} e^{-(r+\lambda)B(\zeta;y)}\max\begin{pmatrix} -\left(\eta e^{\lambda B(\zeta;y)}-1\right)S_{B(\zeta;y)} - K, \\ -\eta e^{\lambda B(\zeta;y)}S_{B(\zeta;y)}\end{pmatrix} I_{B(\zeta;y)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\eta\max\begin{pmatrix} -\left(e^{\lambda\zeta}-1\right)S_\zeta - K, \\ -e^{\lambda\zeta}S_\zeta\end{pmatrix} I_{\zeta<B(\zeta;x)}\end{bmatrix} + K\eta \\
&\ge E^y\begin{bmatrix} e^{-(r+\lambda)B(\zeta;x)}\max\begin{pmatrix} -\left(\eta e^{\lambda B(\zeta;x)}-1\right)S_{B(\zeta;x)} - K, \\ -\eta e^{\lambda B(\zeta;x)}S_{B(\zeta;x)}\end{pmatrix} I_{B(\zeta;x)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\eta\max\begin{pmatrix} -\left(e^{\lambda\zeta}-1\right)S_\zeta - K, \\ -e^{\lambda\zeta}S_\zeta\end{pmatrix} I_{\zeta<B(\zeta;x)}\end{bmatrix} + K\eta \\
&> E^x\begin{bmatrix} e^{-(r+\lambda)B(\zeta;x)}\max\begin{pmatrix} -\left(\eta e^{\lambda B(\zeta;x)}-1\right)S_{B(\zeta;x)} - K, \\ -\eta e^{\lambda B(\zeta;x)}S_{B(\zeta;x)}\end{pmatrix} I_{B(\zeta;x)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\eta\max\begin{pmatrix} -\left(e^{\lambda\zeta}-1\right)S_\zeta - K, \\ -e^{\lambda\zeta}S_\zeta\end{pmatrix} I_{\zeta<B(\zeta;x)}\end{bmatrix} + K\eta \\
&= E^x\begin{bmatrix} e^{-(r+\lambda)B(\zeta;x)}\left(S_{B(\zeta;x)}-K\right)^+ I_{B(\zeta;x)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\eta\left(S_\zeta - K\right)^+ I_{\zeta<B(\zeta;x)}\end{bmatrix} - \eta(x-K) \ge 0.
\end{aligned}$$

We finish the proof by taking $T \to \infty$. □

The corresponding proposition for the buyer's boundary cannot be proven similarly to Proposition 3.3. We shall use a different approach. First, note that Proposition 3.7 from Zaevski [23] can be rewritten as

Proposition 3.4. *Let $\eta = 1$ and $x > K$. Then it is optimal for one of the buyer or the seller to exercise the option immediately.*

Proposition 3.5. *Suppose that $x \in \Upsilon^b$ and $y > x$. Then $y \in \Upsilon^b$ too.*

Proof. Suppose that the proposition is not true, i.e. there exists $y > x$ such that $y \notin \Upsilon^b$. If the value y is seller's optimal, Proposition 3.3 shows that all values below y are also seller's optimal. This can not be true because $x \in \Upsilon^b$ and $x < y$. Therefore $y \in \overline{\Upsilon}$. Note that $\{+\infty\} \in \Upsilon^b$. So, a point from the continuation region, y, is between two points from the buyer's exercise region (x and $+\infty$). Therefore there exist constants B and C, $x < B < y < C$, such that they are buyer's optimal – $\{B, C\} \in \Upsilon^b$, the interval between them is from the continuation region – $(B, C) \in \overline{\Upsilon}$, and $y \in (B, C)$. Let the starting point be y and ζ_B and ζ_C be the first hitting times to the values B and C, respectively. This means that the buyer's optimal stopping time is $\zeta_B \wedge \zeta_C$. Also, it is not optimally for the seller to exercise before $\zeta_B \wedge \zeta_C$. Hence

$$y - K < E^y\left[e^{-(r+\lambda)(\zeta_B\wedge\zeta_C)}\left(S_{\zeta_B\wedge\zeta_C} - K\right)\right]. \tag{8}$$

Let us examine a new game option without penalty, i.e. $\eta = 1$, and the same other parameters. We shall denote by Υ_1^b, Υ_1^s, and $\overline{\Upsilon}_1$ its regions and by $A_1(\cdot)$ and $B_1(\cdot)$ the corresponding optimal stopping times. Suppose that $y \in \Upsilon_1^s$. Since $x < y$, Proposition 3.3 gives us that $x \in \Upsilon_1^s$ too. On the other hand $x \in \Upsilon^b$. This means that it is optimal for the buyer to exercise immediately, provided that the seller can cancel the contract paying a positive penalty. This means that if the seller's penalty is zero, then it is optimal for the buyer to exercise immediately too. Hence $x \in \Upsilon_1^b$. This contradicts $x \in \Upsilon_1^s$ since the sets Υ_1^s and Υ_1^b are disjoint. Hence $y \notin \Upsilon_1^s$. Note that this is true for all points from the interval (B, C).

Suppose now that $y \in \Upsilon_1^b$. Above we proved that there is not seller's optimal points in the interval (B, C). Hence, if $\zeta = \zeta_B \wedge \zeta_C$, then $A_1(\zeta, y) > \zeta$. Therefore

$$y - k \ge E^y\left[e^{-(r+\lambda)(\zeta\wedge A_1(\zeta_C;y))}\left(S_{\zeta\wedge A_1(\zeta;y)} - K\right)\right] = E^y\left[e^{-(r+\lambda)(\zeta_C\wedge\zeta_B)}\left(S_{\zeta_C\wedge\zeta_B} - K\right)\right],$$

which contradicts inequality (8). Hence $y \notin \Upsilon_1^b$. Therefore $y \in \overline{\Upsilon}_1$. But this contradicts Proposition 3.4, which says that $\overline{\Upsilon}_1$ is empty when $\eta = 1$. Therefore $y \in \Upsilon^b$. □

Remark 3.1. Actually, the proof is based on the fact that if a part of the continuation region is between two points of the buyer's optimal region, then the option payment structure is the same as the payment structure of a game option with $\eta = 1$.

The following proposition provides the fact that if the risk free rate is negative, then the seller's exercise region consists only of the points below the strike.

Proposition 3.6. *If $r < 0$, then $\Upsilon^s = (0, K]$.*

Proof. Suppose that x is larger than the strike, $x > K$, $x \in \Upsilon^s$ and it is not an exercise boundary. Proposition 3.5 gives us that there exists a constant $K_1 > x$, such that $K_1 \notin \Upsilon^b$ too. Let us define ζ as the first exit time from the strip (K, K_1). Proposition 3.5 shows also that $B(\zeta; x) > \zeta$. Hence

$$E^x\left[e^{-(r+\lambda)\zeta}\eta\left(S_\zeta - K\right)I_{\zeta \le B(\zeta;x)} + e^{-(r+\lambda)B(\zeta;x)}\left(S_\zeta - K\right)I_{B(\zeta;x)<\zeta}\right] \le E^x\left[e^{-r\zeta}\eta\left(S_\zeta - K\right)\right] < \eta(x-K),$$

which shows that $x \notin \Upsilon^s$.[1] □

3.2. Put style options

Now we shall prove the analogues of the propositions in Section 3.1. First, note that all points above the strike are optimal for the seller, because he has to pay nothing.

Proposition 3.7. *If $x \ge K$, then $x \in \Upsilon^s$.*

Proposition 3.8. *Suppose that $x \in \Upsilon^s$ and $x < y$. Then $y \in \Upsilon^s$ too.*

Proof. Obviously, we have to examine only the case $x < y < K$. Let T be a fixed and finite moment. Let $\zeta \in \mathcal{T}[0, T]$ be an arbitrary stopping time and $B(\zeta; x)$ be the corresponding ζ-buyer's optimal strategy. Then condition (6) leads to

$$E^x\left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)}\left(K - S_{B(\zeta;x)}\right)^+ I_{B(\zeta;x)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\eta\left(K - S_\zeta\right)^+ I_{\zeta<B(\zeta;x)} \end{array}\right] \ge \eta(K-x).$$

Analogously to the proof of Proposition 3.3, we derive

$$\begin{aligned}
&E^y\left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;y)}\left(K - S_{B(\zeta;y)}\right)^+ I_{B(\zeta;y)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\eta\left(K - S_\zeta\right)^+ I_{\zeta<B(\zeta;y)} \end{array}\right] - \eta(K-y) \\
&= E^y\left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;y)}\left(K - S_{B(\zeta;y)}\right)^+ I_{B(\zeta;y)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\eta\left(K - S_\zeta\right)^+ I_{\zeta<B(\zeta;y)} \end{array}\right] \\
&\quad + \eta E^y\left[e^{-(r+\lambda)B(\zeta;y)}e^{\lambda B(\zeta;y)}S_{B(\zeta;y)}I_{B(\zeta;y)\le\zeta} + e^{-(r+\lambda)\zeta}e^{\lambda\zeta}S_\zeta I_{\zeta<B(\zeta;y)}\right] - K\eta \\
&= E^y\left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;y)}\max\begin{pmatrix}\left(\eta e^{\lambda B(\zeta;y)} - 1\right)S_{B(\zeta;y)} + K, \\ \eta e^{\lambda B(\zeta;y)}S_{B(\zeta;y)}\end{pmatrix} I_{B(\zeta;y)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\eta\max\begin{pmatrix}\left(e^{\lambda\zeta} - 1\right)S_\zeta + K, \\ e^{\lambda\zeta}S_\zeta\end{pmatrix} I_{\zeta<B(\zeta;x)} \end{array}\right] - K\eta \\
&\ge E^y\left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)}\max\begin{pmatrix}\left(\eta e^{\lambda B(\zeta;x)} - 1\right)S_{B(\zeta;x)} + K, \\ \eta e^{\lambda B(\zeta;x)}S_{B(\zeta;x)}\end{pmatrix} I_{B(\zeta;x)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\eta\max\begin{pmatrix}\left(e^{\lambda\zeta} - 1\right)S_\zeta + K, \\ e^{\lambda\zeta}S_\zeta\end{pmatrix} I_{\zeta<B(\zeta;x)} \end{array}\right] - K\eta \\
&> E^x\left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)}\max\begin{pmatrix}\left(\eta e^{\lambda B(\zeta;x)} - 1\right)S_{B(\zeta;x)} + K, \\ \eta e^{\lambda B(\zeta;x)}S_{B(\zeta;x)}\end{pmatrix} I_{B(\zeta;x)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\eta\max\begin{pmatrix}\left(e^{\lambda\zeta} - 1\right)S_\zeta + K, \\ e^{\lambda\zeta}S_\zeta\end{pmatrix} I_{\zeta<B(\zeta;x)} \end{array}\right] - K\eta \\
&= E^x\left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)}\left(K - S_{B(\zeta;x)}\right)^+ I_{B(\zeta;x)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\eta\left(K - S_\zeta\right)^+ I_{\zeta<B(\zeta;x)} \end{array}\right] - \eta(K-x) \ge 0.
\end{aligned}$$

After taking $T \to \infty$ we finish the proof. □

Note that Proposition 3.4 still holds. Its proof is close to the proof of Proposition 3.7 from Zaevski [23]. Also, the following proposition holds for the buyer's exercise boundary, which proof is the same as the proof of Proposition 3.5.

[1] We used that the discounted asset price is a martingale.

Proposition 3.9. *Suppose that* $x \in \Upsilon^b$ *and* $0 < y < x$. *Then* $y \in \Upsilon^b$ *too.*

Finally, we shall prove the analogue of Proposition 3.6.

Proposition 3.10. *If* $r > 0$, *then* $\Upsilon^s = [K, \infty)$.

Proof. Suppose that $x < K$, $x \in \Upsilon^s$, and x is not the exercise boundary. Using Proposition 3.9 we see that there exists a constant $K_1 < x$, such that $K_1 \notin \Upsilon^b$. Let ζ be defined as the first exit time from the strip (K_1, K). Proposition 3.9 leads also to $B(\zeta; x) > \zeta$. Thus we have

$$E^x\left[e^{-(r+\lambda)\zeta}\eta\left(K-S_\zeta\right)I_{\zeta \le B(\zeta;x)} + e^{-(r+\lambda)B(\zeta;x)}\left(K-S_\zeta\right)I_{B(\zeta;x)<\zeta}\right] \le E^x\left[e^{-r\zeta}\eta\left(K-S_\zeta\right)\right] < \eta(K-x).$$

Therefore $x \notin \Upsilon^s$. □

3.3. Some notes on the finite maturity case

Although the purpose of this article is to study game options without a maturity restriction, most of the propositions proven above are true when the maturity is finite, $T < \infty$. Unfortunately, we can not expect in this case that the exercise boundaries are flat. However, the obtained propositions give us some hints for their form.

Let us consider the call style options. First, Proposition 3.2 states that all points bellow the strike are seller's optimal. Also, if the current asset value is larger than the strike and the time to maturity is small enough, then it is better for the seller to keep the option till the maturity. This means that the seller's exercise boundary is the strike near the maturity. If the penalty coefficient is small enough and the risk-free rate is positive (see Proposition 3.6), we may expect that the seller's exercise boundary is larger than the strike for large values of the time to maturity. However, in both cases Proposition 3.3 shows that all points bellow the seller's exercise boundary are optimal to him/her. If the discount rate is zero, Proposition 3.1 says that it is never optimal for the buyer to exercise. Otherwise, if $\lambda > 0$ Proposition 3.5 provides that all points above the buyer's exercise boundary are buyer's optimal.

The put style options lead to, in some sense, symmetric exercise regions. All points above the strike are seller's optimal and near the maturity they are the unique such points. Proposition 3.10 indicates that when the risk free rate is positive, again only the points above the strike form the seller's optimal region. Otherwise, if $r < 0$ and the penalty coefficient is small enough there has to exist a critical value for the time to maturity after which the seller's exercise boundary is bellow the strike. However, regardless of the seller's boundary value, all points above it are seller's optimal – see Proposition 3.8. Also, Proposition 3.9 hints us that there exists a boundary bellow which early exercising is optimal for the buyer.

4. Pricing of discounted game call options

Suppose that the asset price is described by the log-normal process

$$dS_t = rS_t + \sigma S_t dB_t. \tag{9}$$

Propositions 3.3, 3.2, and 3.5 prompt us to expect that the exercise regions have the form $\Upsilon^s = (0, A]$ and $\Upsilon^b = (B, \infty)$, $K \le A \le B$. As we shall see later it is possible both boundaries to be equal to the strike. This explains why the seller's optimal region is closed, whereas the buyer's one is open. Note that the strike can not be buyer's optimal, because the option payment is zero. The same reason implies that the strike is always seller's optimal. Suppose that the initial asset price belongs to the continuation region $A < x \le B$ and τ_A and τ_B are the first hitting times of the asset price to the boundaries A and B. They can be described also as the first hitting times of the Brownian motion with drift

$$\psi = \frac{r}{\sigma} - \frac{\sigma}{2} \tag{10}$$

to the values

$$\begin{aligned} \overline{A} &= \frac{\ln A - \ln x}{\sigma} < 0 \\ \overline{B} &= \frac{\ln B - \ln x}{\sigma} > 0. \end{aligned} \tag{11}$$

Thus the end of the option life can be viewed as $\tau = \tau^A \wedge \tau^B$. Let us define the following variables

$$\begin{aligned} \mu &:= \frac{\psi}{\sigma}, \\ c &:= \sqrt{\mu^2 + 2\frac{r+\lambda}{\sigma^2}} \equiv \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}}, \end{aligned} \tag{12}$$

$q := c + \mu$, and $p := 2c$. Since $\lambda \ge 0$, we have

$$c = \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} \ge \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r}{\sigma^2}} \ge \frac{r}{\sigma^2} + \frac{1}{2} = \mu + 1.$$

This is equivalent to $p \geq q+1$. Note that $r > 0$ when $\lambda = 0$, since $r+\lambda > 0$. Therefore $p = q+1$ when $\lambda = 0$. We can see also that the following lemma holds.

Lemma 4.1. *We have that the inequality $2q+1 > p$ is equivalent to $r > 0$.*

Using formulas (A.3) and (A.4) for the Laplace transforms of the first exit time of the Brownian motion from a strip[2] we derive that the price of a game call option is

$$\begin{aligned} f(A,B,x) &= E\begin{bmatrix} e^{-(r+\lambda)\tau^B}(S_{\tau^B}-K)I_{\tau^B\leq\tau^A} \\ +e^{-(r+\lambda)\tau^A}\eta(S_{\tau^A}-K+\eta)I_{\tau^A<\tau^B} \end{bmatrix} \\ &= (B-K)E^x\left[e^{-(r+\lambda)\tau^B}I_{\tau^B\leq\tau^A}\right]+\eta(A-K)E^x\left[e^{-(r+\lambda)\tau^A}I_{\tau^A<\tau^B}\right]. \\ &= \eta(A-K)e^{\psi\bar{A}}\frac{\sinh\left(\sigma c\bar{B}\right)}{\sinh\left(\sigma c\left(\bar{B}-\bar{A}\right)\right)}+(B-K)e^{\psi\bar{B}}\frac{\sinh\left(-\sigma c\bar{A}\right)}{\sinh\left(\sigma c\left(\bar{B}-\bar{A}\right)\right)} \\ &= \eta(A-K)\left(\frac{A}{x}\right)^q\frac{B^p-x^p}{B^p-A^p}+(B-K)\left(\frac{B}{x}\right)^q\frac{x^p-A^p}{B^p-A^p}. \end{aligned} \tag{13}$$

Now we shall fix one of the boundaries A/B and then we shall derive which value of B/A maximizes/minimizes function (13). First, suppose that B is fixed and let us change the variables as $a=\frac{A}{B}$, $k=\frac{K}{B}$, and $y=\frac{x}{B}$. Hence $0<k<a<1$. We can write Eq. (13) as

$$f(A,B,x)=\frac{B}{y^q}\frac{\eta(a-k)a^q(1-y^p)+(1-k)(y^p-a^p)}{1-a^p}. \tag{14}$$

We have to find which value of the variable a minimizes the function

$$g(a)=\frac{\eta(a-k)a^q(1-y^p)+(1-k)(y^p-a^p)}{1-a^p}. \tag{15}$$

Its first derivative is

$$g_a(a)=\frac{1-y^p}{(1-a^p)^2}a^{q-1}\begin{bmatrix} a^{p+1}\eta(p-q-1)-a^p\eta k(p-q) \\ -a^{p-q}p(1-k)+a(q+1)\eta-q\eta k \end{bmatrix}. \tag{16}$$

We prove in Appendix B.1 that it has just one root in the interval $(0, 1)$. It leads to a minimum for option price (14), which we shall denote by $a(B)$. Note that it is independent of the variable y, respectively x. Thus we conclude that if the buyer's strategy is the first hitting moment to the value B, then the seller's strategy is the first hitting to the value $Ba(B)$.

We proceed analogously for the buyer's boundary. Let us fix A and change variables as $b=\frac{B}{A}$, $k=\frac{K}{A}$, and $y=\frac{x}{A}$. Thus price (13) turns to

$$f(A,B,x)=\frac{A}{y^q}\frac{\eta(1-k)(b^p-y^p)+(b-k)b^q(y^p-1)}{b^p-1}. \tag{17}$$

Let the function $g(\cdot)$ be defined as

$$g(b)=\frac{A}{y^q}\frac{\eta(1-k)(b^p-y^p)+(b-k)b^q(y^p-1)}{b^p-1}. \tag{18}$$

and it has a derivative

$$g_b(b;y)=\frac{y^p-1}{(b^p-1)^2}b^{q-1}\begin{bmatrix} -b^{p+1}(p-q-1)+b^pk(p-q) \\ +b^{p-q}p(1-k)\eta-b(q+1)+qk \end{bmatrix}. \tag{19}$$

We shall examine now the case when discounting really exists, i.e. $\lambda > 0$ and therefore $p > q+1$. The undiscounted case is studied in the following section. In Appendix B.2 is proven that this derivative has just one zero point in the interval $(1, \infty)$ when $A > K$, or when $A = K$ and $r > 0$. It leads to a maximum for function (18). We shall denote it by $b(A)$. Thus we can conclude that if the seller's strategy is first hitting to the value A, then the buyer's strategy is the first hitting moment to $Ab(A)$. Thus we can derive our candidate $\overline{A}$ for the seller's boundary as the solution of the equation

$$b(A)a(Ab(A))=1. \tag{20}$$

The buyer's boundary candidate is $\overline{B}=b\left(\overline{A}\right)\overline{A}$. Note that we avoid a non-differentiability in price function (13) by changing the payment $\eta(A-k)^+$ to $\eta(A-k)$. Hence, it may happen that $\overline{A}<K$. Proposition 3.2 says that all points below the strike are optimal for the seller. Thus we conclude that the true seller's boundary is

$$A^*=\max\left(K,\overline{A}\right). \tag{21}$$

[2] (3.0.5 a & b) from Borodin and Salminen [2].

We know from Proposition 3.6 that the true seller's boundary is $A^* = K$ when $r \leq 0$. In Appendix B.2 is shown that in that case, $A^* = K$ and $r \leq 0$, the derivative (19) is negative and therefore the buyer's exercise region is $\Upsilon^b = (K, \infty)$. Otherwise, if $r > 0$ and $A^* = K$, then the buyer's exercise boundary is larger than the strike and it is obtained as

$$B^* = A^* b(A^*). \tag{22}$$

Now we can formulate the following theorem.

Theorem 4.1. *Let the discount rate be positive, $\lambda > 0$.*

1. *Suppose that $r > 0$. Then the seller's and buyer's boundaries are A^* and B^*, defined in Eqs. (21) and (22). $\overline{A}$ is the solution of Eq. (20). Thus the game call option price Y is*
   (a) *If $x \leq A^*$, then $Y = \eta(x-K)^+$.*
   (b) *If $A^* < x < B^*$, then*
   $$Y = \eta(A^* - K)\left(\frac{A^*}{x}\right)^q \frac{B^{*p} - x^p}{B^{*p} - A^{*p}} + (B^* - K)\left(\frac{B^*}{x}\right)^q \frac{x^p - A^{*p}}{B^{*p} - A^{*p}}. \tag{23}$$
   (c) *If $B^* \leq x$, then $Y = x - K$.*
2. *If $r \leq 0$, then $\Upsilon^s = (0, K]$ and $\Upsilon^b = (K, \infty)$. Thus we can conclude that if $x \leq K$, then $Y = 0$, and $Y = x - K$, otherwise.*

5. Perpetual game call options without discounting

Let us consider the undiscounted case, $\lambda = 0$. Note that the risk free rate has to be positive, $r > 0$. Proposition 3.1 gives us that the early exercise is never optimal for the buyer. The same can be derived using Appendix B.2. There is proven that the derivative (19) is positive in the interval $(1, \infty)$ and therefore price function (17) has a maximum for $b = \infty$. We suppose again that the seller's optimal strategy is the first hitting time to the barrier $A \geq K$. Let this stopping time be ζ. In that case the price of the game call option can be written as

$$\begin{aligned} Y &= E^x\left[\eta e^{-r\zeta}\left(S_\zeta - K\right)^+ I_{\zeta<\infty}\right] + \lim_{T\to\infty} E^x\left[e^{-rT}(S_T - K)^+ I_{T<\zeta}\right] \\ &= \eta(A-K)E^x\left[e^{-r\zeta} I_{\zeta<\infty}\right] + \lim_{T\to\infty} C_{DO}(x, A, K), \end{aligned} \tag{24}$$

where $C_{DO}(x, A, K)$ is the price of a down-and-out barrier option with the strike K and the barrier A. Using Rubinstein [18] we can see that

$$\lim_{T\to\infty} C_{DO}(x, A, K) = x\left(1 - \left(\frac{A}{x}\right)^{1+\frac{2r}{\sigma^2}}\right). \tag{25}$$

Using Eq. (A.2) from Proposition A.1 we find

$$E^x\left[e^{-r\zeta} I_{\zeta<\infty}\right] = \left(\frac{A}{x}\right)^{\frac{2r}{\sigma^2}}. \tag{26}$$

Combining Eqs. (25) and (26) we see that Eq. (24) turns to

$$Y(A) = x + \left(\frac{A}{x}\right)^{\frac{2r}{\sigma^2}} (A(\eta - 1) - \eta K). \tag{27}$$

Calculating the A-derivative of function (27) we find

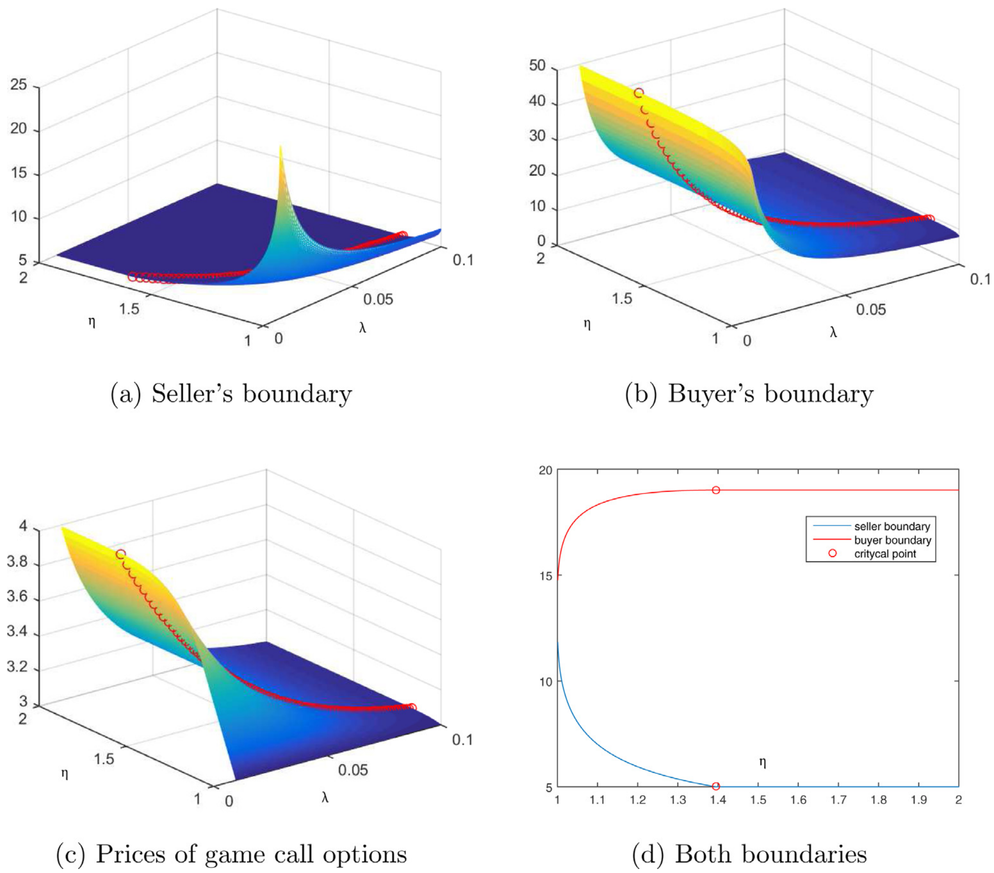
$$Y'(A) = \left(\frac{A}{x}\right)^{\frac{2r}{\sigma^2}} \left((\eta - 1)\frac{2r + \sigma^2}{\sigma^2} - \frac{2r\eta K}{\sigma^2 A}\right). \tag{28}$$

Let us define $\overline{\eta}$ as

$$\overline{\eta} = \frac{2r + \sigma^2}{\sigma^2}. \tag{29}$$

Obviously, $Y'(A)$ is an increasing function. If we suppose that $\overline{\eta} < \eta$, then $Y'(A) > 0$ for all $A \geq K$ and therefore function (27) has a minimum for $A = K$. In that case the seller's optimal strategy is the first hitting time to the strip $(0, K]$. The option price is

$$Y = x - K\left(\frac{K}{x}\right)^{\frac{2r}{\sigma^2}} \tag{30}$$



(a) Seller's boundary (b) Buyer's boundary

(c) Prices of game call options (d) Both boundaries

Fig. 1. Call options.

when $x > K$, and zero otherwise. Alternatively, if we suppose $1 < \eta \leq \overline{\eta}$ we see that the function $Y(A)$ first decreases and then increases. Therefore the optimal seller's strategy is the first hitting time to the the strip $\left(0, \overline{A}\right]$ where $\overline{A}$ makes derivative (28) to be zero,

$$\overline{A} = \frac{2r\eta K}{(\eta-1)\left(2r+\sigma^2\right)}. \tag{31}$$

Hence, if the initial asset value x is below the barrier $\overline{A}$, immediate exercising is optimal and therefore the option price is $Y = \eta(x-K)^+$. Otherwise, if $x > \overline{A}$, then the seller's optimal strategy is the first hitting time to $\overline{A}$, which leads to the option price

$$Y = x - \frac{\eta K \sigma^2}{\left(2r+\sigma^2\right)} \left(\frac{\overline{A}}{x}\right)^{\frac{2r}{\sigma^2}}. \tag{32}$$

We can summarize these results in the following theorem.

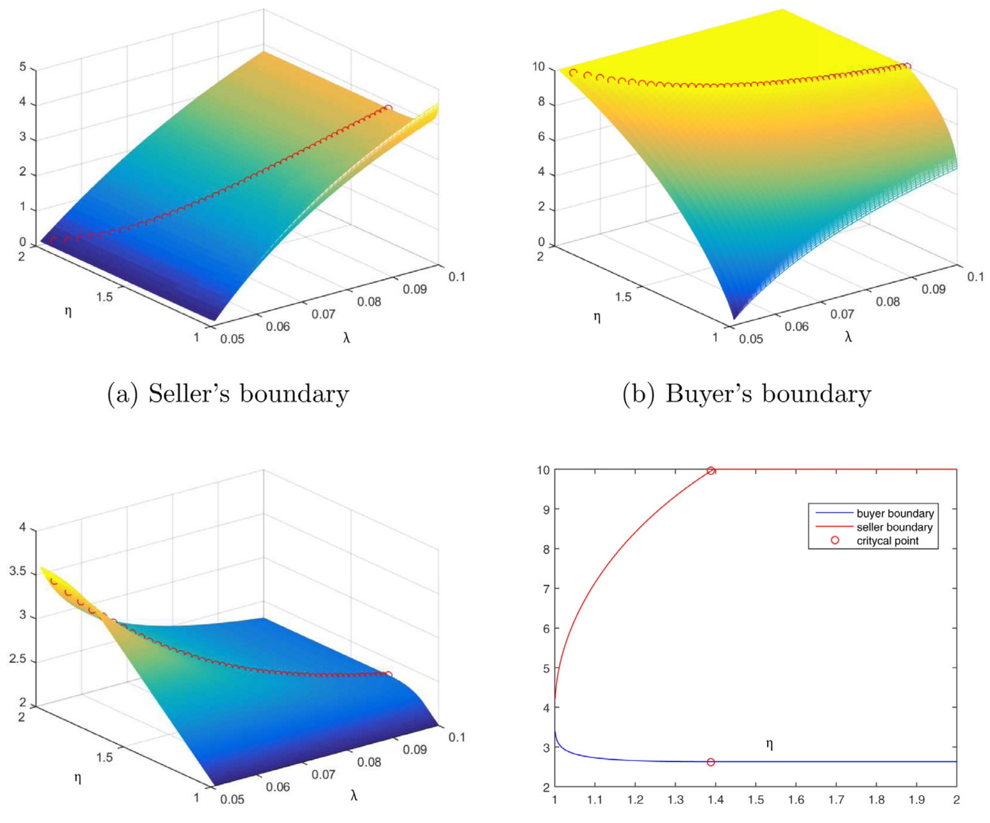
Theorem 5.1. *If $\lambda = 0$, then $\Upsilon^b = \emptyset$. Also*

1. *If $\overline{\eta} < \eta$, where $\overline{\eta}$ is defined in (29), then the seller's exercise region is $\Upsilon^s = (0, K]$. When $x > K$, the option price is given by Eq. (30). Otherwise it is zero, $Y = 0$.*
2. *If $1 < \eta \leq \overline{\eta}$, then the seller's exercise region is $\Upsilon^s = \left(0, \overline{A}\right]$, where $\overline{A}$ is given by Eq. (31). Thus if $x > \overline{A}$, then the option price is given by Eq. (32). Otherwise, it is $Y = \eta(x-K)^+$.*

This result is alternatively derived in Ekström and Villeneuve [6].

6. Pricing put style options

The payments for a put style option are defined by Eq. (2). We shall proceed as in Section 4. We shall emphasize the differences between the call and put style options. Propositions 3.7, 3.8, and 3.9 indicate that the seller's and buyer's optimal



(a) Seller's boundary (b) Buyer's boundary

(c) Prices of game put option (d) Both boundaries

Fig. 2. Put options.

regions have to be of the form $\Upsilon^s = [B, \infty)$ and $\Upsilon^b = (0, A)$, respectively, where $A < B \le K$. Suppose that $A < x < B$. The analogue of call price function (13) is

$$f(A,B,x) = (A-K)\left(\frac{A}{x}\right)^q \frac{B^p - x^p}{B^p - A^p} + \eta(B-K)\left(\frac{B}{x}\right)^q \frac{x^p - A^p}{B^p - A^p}. \tag{33}$$

Suppose that $A < K$ is fixed. After changing the variables $b = \frac{B}{A}$, $k = \frac{K}{A}$, and $y = \frac{x}{A}$, the b-derivative of price function (33) turns to

$$g_b(b;y) = \frac{y^p - 1}{(b^p - 1)^2} b^{q-1} \begin{bmatrix} b^{p+1}\eta(p-q-1) - b^p \eta k(p-q) \\ +b^{p-q}p(k-1) + b(q+1)\eta - q\eta k \end{bmatrix}. \tag{34}$$

Suppose that $\lambda > 0$ or equivalently $p > q+1$. We shall comment the undiscounted case $\lambda = 0$ later. In Appendix B.4 is proven that the equation

$$b^{p+1}\eta(p-q-1) - b^p \eta k(p-q) + b^{p-q}p(k-1) + b(q+1)\eta - q\eta k \tag{35}$$

has a unique solution larger than one. We shall denote it by $b(A)$ and it minimizes price function (33).

Let now the boundary $B \le K$ be fixed. We change variables to $a = \frac{A}{B}$, $k = \frac{K}{B}$, and $y = \frac{x}{B}$. Hence, the a-derivative of price function (33) is

$$g_a(a) = \frac{1 - y^p}{(1 - a^p)^2} a^{q-1} \begin{bmatrix} -a^{p+1}(p-q-1) + a^p k(p-q) \\ -a^{p-q}p(k-1)\eta - a(q+1) + qk \end{bmatrix}. \tag{36}$$

In (B.3) is proven that if $B < K$, then the equation

$$-a^{p+1}(p-q-1) + a^p k(p-q) - a^{p-q}p(k-1)\eta - a(q+1) + qk = 0 \tag{37}$$

has a unique solution less than one. It leads to a maximum for price function (33). We shall denote it by $a(B)$. Hence, if the seller's strategy is the first hitting time to B, then the buyer's optimal strategy is the first hitting time to $Ba(B)$. Thus our

Table 1
Call option prices and optimal boundaries.

	A^*	B^*	$S_0 = 6$	$S_0 = 7$	$S_0 = 8$	$S_0 = 9$
penalty cefficient $\eta = 1.001$						
$\lambda = 0.001$	157.1248	349.5925	1.0010/s	2.0020/s	3.0030/s	4.0040/s
$\lambda = 0.01$	24.6603	35.1478	1.0010/s	2.0020/s	3.0030/s	4.0040/s
$\lambda = 0.1$	7.0278	7.9334	1.0010/s	2.0020/s	3.0000/b	4.0000/b
$\lambda = 0.15$	6.3311	6.9701	1.0010/s	2.0000/b	3.0000/b	4.0000/b
$\lambda = 0.2$	5.9832	6.4884	1.0010/c	2.0000/b	3.0000/b	4.0000/b
penalty cefficient $\eta = 1.2$						
$\lambda = 0.001$	13.0937	472.2606	1.2000/s	2.4000/s	3.6000/s	4.8000/s
$\lambda = 0.01$	7.9555	48.4716	1.2000/s	2.4000/s	3.5999/c	4.7561/c
$\lambda = 0.1$	5.0000	8.9543	1.0740/c	2.0531/c	3.0160/c	4.0000/b
$\lambda = 0.15$	5.0000	7.5978	1.0356/c	2.0082/c	3.0000/b	4.0000/b
$\lambda = 0.2$	5.0000	6.9322	1.0170/c	2.0000/b	3.0000/b	4.0000/b
penalty cefficient $\eta = 1.5$						
$\lambda = 0.001$	7.2323	475.8458	1.5000/s	3.0000/s	4.4581/c	5.8037/c
$\lambda = 0.01$	5.5373	49.5742	1.4740/c	2.7808/c	3.9641/c	5.0703/c
$\lambda = 0.1$	5.0000	8.9543	1.0740/c	2.0531/c	3.0160/c	4.0000/b
$\lambda = 0.15$	5.0000	7.5978	1.0356/c	2.0082/c	3.0000/b	4.0000/b
$\lambda = 0.2$	5.0000	6.9322	1.0170/c	2.0000/b	3.0000/b	4.0000/b
penalty cefficient $\eta = 1.7$						
$\lambda = 0.001$	5.9906	476.3748	1.7000/c	3.2865/c	4.7086/c	6.0231/c
$\lambda = 0.01$	5.0000	49.6448	1.5084/c	2.8092/c	3.9882/c	5.0911/c
$\lambda = 0.1$	5.0000	8.9543	1.0740/c	2.0531/c	3.0160/c	4.0000/b
$\lambda = 0.15$	5.0000	7.5978	1.0356/c	2.0082/c	3.0000/b	4.0000/b
$\lambda = 0.2$	5.0000	6.9322	1.0170/c	2.0000/b	3.0000/b	4.0000/b
penalty cefficient $\eta = 2$						
$\lambda = 0.001$	5.0260	476.5680	1.8265/c	3.3929/c	4.8002/c	6.1034/c
$\lambda = 0.01$	5.0000	49.6448	1.5084/c	2.8092/c	3.9882/c	5.0911/c
$\lambda = 0.1$	5.0000	8.9543	1.0740/c	2.0531/c	3.0160/c	4.0000/b
$\lambda = 0.15$	5.0000	7.5978	1.0356/c	2.0082/c	3.0000/b	4.0000/b
$\lambda = 0.2$	5.0000	6.9322	1.0170/c	2.0000/b	3.0000/b	4.0000/b

candidate for the seller's boundary $\overline{B}$ is the solution of the equation

$$1 = a(B)b(Ba(B)). \tag{38}$$

Proposition 3.7 shows that the true boundary is given by

$$B^* = \min\left(\overline{B}, K\right). \tag{39}$$

Suppose that $r \geq 0$. Proposition 3.10 gives us that $\Upsilon^s = [K, \infty)$ or equivalently $B^* = K$. On the other hand in Appendix B.3 we prove that if the seller's boundary is the strike and $r \geq 0$, then the derivative (36) is positive. Hence, the buyer's exercise region is $\Upsilon^b = (0, K)$.

We omitted above the case $\lambda = 0$ for the seller's boundary. In that case we have $r > 0$, since $r + \lambda > 0$ and thus all we say above is valid in this case too. Also, note that in Appendix B.4 we prove that when $\lambda = 0$, derivative (34) is negative in the whole interval $(1, \infty)$. Hence, the price function (33) has minimum for $\overline{B} = \infty$ and therefore $B^* = K$ always when $\lambda = 0$.

Otherwise, if $r < 0$, then the boundary

$$A^* = B^* a(B) \tag{40}$$

really exists and $A^* < B^* \leq K$. We summarize all these results in the following theorem.

Theorem 6.1.

1. *If $r \geq 0$, then the exercise regions are $\Upsilon^s = [K, \infty)$ and $\Upsilon^b = (0, K)$. If $x < K$, then the option price is $Y = K - x$ and it is zero, otherwise.*
2. *If $r < 0$, then the exercise regions are $\Upsilon^s = [B^*, \infty)$ and $\Upsilon^b = (0, A^*)$, where B^* and A^* are given by Eqs. (39) and (40), respectively. The value of $\overline{B}$ is obtained as the solution of Eq. (38). Thus the option price is*
   (a) $Y = K - x$, *if* $x < A^*$;
   (b) $Y = (A^* - K)\left(\frac{A^*}{x}\right)^q \frac{B^{*p} - x^p}{B^{*p} - A^{*p}} + \eta (B^* - K)\left(\frac{B^*}{x}\right)^q \frac{x^p - A^{*p}}{B^{*p} - A^{*p}}$, *if* $x \in [A^*, B^*)$;
   (c) $Y = \eta (K - x)^+$, *if* $B^* \leq x$.

Table 2
Put option prices and optimal boundaries.

	A^*	B^*	$S_0 = 5$	$S_0 = 6$	$S_0 = 7$	$S_0 = 8$
penalty cefficient $\eta = 1.001$						
$\lambda = 0.051$	0.1430	0.3182	5.0050/s	4.0040/s	3.0030/s	2.0020/s
$\lambda = 0.06$	1.4226	2.0275	5.0050/s	4.0040/s	3.0030/s	2.0020/s
$\lambda = 0.1$	4.6118	5.4933	5.0020/c	4.0040/s	3.0030/s	2.0020/s
$\lambda = 0.15$	6.3026	7.1146	5.0000/b	4.0000/b	3.0028/c	2.0020/s
$\lambda = 0.2$	7.1735	7.8975	5.0000/b	4.0000/b	3.0000/b	2.0020/s
penalty cefficient $\eta = 1.2$						
$\lambda = 0.051$	0.1059	3.8186	6.0000/s	4.8000/s	3.6000/s	2.4000/s
$\lambda = 0.06$	1.0315	6.2850	5.8511/c	4.7925/c	3.6000/s	2.4000/s
$\lambda = 0.1$	3.8070	9.3291	5.0857/c	4.2015/c	3.2836/c	2.2904/c
$\lambda = 0.15$	5.5839	10.0000	5.0000/b	4.0086/c	3.0667/c	2.1177/c
$\lambda = 0.2$	6.5809	10.0000	5.0000/b	4.0000/b	3.0069/c	2.0481/c
penalty cefficient $\eta = 1.5$						
$\lambda = 0.051$	0.1051	6.9134	7.0688/c	5.9010/c	4.5000/s	3.0000/s
$\lambda = 0.06$	1.0086	9.0296	6.1262/c	5.2078/c	4.1384/c	2.9056/c
$\lambda = 0.1$	3.7952	10.0000	5.0888/c	4.2081/c	3.2946/c	2.3068/c
$\lambda = 0.15$	5.5839	10.0000	5.0000/b	4.0086/c	3.0667/c	2.1177/c
$\lambda = 0.2$	6.5809	10.0000	5.0000/b	4.0000/b	3.0069/c	2.0481/c
penalty cefficient $\eta = 1.7$						
$\lambda = 0.051$	0.1050	8.3464	7.2638/c	6.1880/c	4.8972/c	3.3865/c
$\lambda = 0.06$	1.0072	10.0000	6.1444/c	5.2353/c	4.1772/c	2.9579/c
$\lambda = 0.1$	3.7952	10.0000	5.0888/c	4.2081/c	3.2946/c	2.3068/c
$\lambda = 0.15$	5.5839	10.0000	5.0000/c	4.0086/c	3.0667/c	2.1177/c
$\lambda = 0.2$	6.5809	10.0000	5.0000/b	4.0000/b	3.0069/c	2.0481/c
penalty cefficient $\eta = 2$						
$\lambda = 0.051$	0.1049	9.9484	7.3351/c	6.2931/c	5.0429/c	3.5799/c
$\lambda = 0.06$	1.0072	10.0000	6.1444/c	5.2353/c	4.1772/c	2.9579/c
$\lambda = 0.1$	3.7952	10.0000	5.0888/c	4.2081/c	3.2946/c	2.3068/c
$\lambda = 0.15$	5.5839	10.0000	5.0000/c	4.0086/c	3.0667/c	2.1177/c
$\lambda = 0.2$	6.5809	10.0000	5.0000/b	4.0000/b	3.0069/c	2.0481/c

7. Numerical results

In Fig. 1 we present the behavior of discounted game call options. We assume that the risk free rate is $r = 0.05$, the volatility is $\sigma = 0.3$, the strike is $K = 5$, and the initial asset value is $x = 8$. We vary the discount factor between 0.01 and 0.1 and the penalty coefficient η between 1.001 and 2. In Fig. 1a and b are shown the buyer's and seller's boundaries, respectively. The behavior of the option price is presented at Fig. 1c. We mark by red points this values of η above which the seller's boundary is equal to the strike. Note also that the optimal boundaries do not depend on η above these critical values as well as the option prices. In Fig. 1d we commonly presented the buyer's and seller's boundaries for a fixed value of the discount rate, $\lambda = 0.03$. The red point is again the critical value of the penalty coefficient. We have a confirmation of Proposition 3.4. When η tends to one, the boundaries tend each to each other.

In Table 1 we reported some particular prices of game call options as well as the corresponding optimal boundaries. We vary the initial asset value as $S \in \{6;\ 7;\ 8;\ 9\}$, the penalty coefficient as $\eta \in \{1.001;\ 1.2;\ 1.5;\ 1.7;\ 2\}$, and the discount rate as $\lambda \in \{0.001;\ 0.01;\ 0.1;\ 0.15;\ 0.2\}$. The rest parameters are the same. After the corresponding price we report in which region is the initial asset price – "b" for the buyer's region, "s" for the seller's region, and "c" for the continuation region. Proposition 3.1 has a confirmation here – when the discount rate is small, $\lambda = 0.001$, the buyer's boundary is very high. Also, note that when the seller's boundary is equal to the strike, the optimal boundaries and the option prices are independent of the penalty coefficient η. This is due to the fact that if the seller's boundary coincides with the strike, then price function (13) does not depend on η.

We present the behavior of put style options at Fig. 2. The initial asset price, the volatility, and the penalty coefficient are as above. We assume that the strike is $K = 10$. Since more interesting is the case when the risk free rate is negative, we assume that $r = -0.05$. We had impose the condition $r + \lambda > 0$ and therefore we vary the discount rate between 0.051 and 0.1. In Fig. 2a and b are presented the buyer's and seller's boundaries, respectively. By red points are marked this values of the penalty coefficient η above which the seller's boundary is equal to the strike. The game put option prices are plotted at Fig. 2c. We give commonly both boundaries for a fixed value of the discount rate $\lambda = 0.08$. We can see that if η tends to one, both boundaries tends one to other. This confirms Proposition 3.4 in the put case.

Some particular option prices and optimal boundaries for game put options are presented in Table 2. We vary the initial asset value as $S \in \{5;\ 6;\ 7;\ 8\}$ and the discount rate as $\lambda \in \{0.051;\ 0.06;\ 0.1;\ 0.15;\ 0.2\}$. The penalty coefficient belongs to the set {1.001; 1.2; 1.5; 1.7; 2}. We again use values $r = -0.05$, $K = 10$, $\sigma = 0.3$. We also report in which region the corresponding initial asset value belongs. We can see that when the seller's boundary is equal to the strike, the option prices and optimal boundaries are independent of the penalty coefficient η. The reasons are the same as in the call case.

8. Conclusions and further work

In this paper we had examined a perpetual game option pricing problem in the Black-Scholes framework. We introduced a discount factor which determines the time structure of the option payments. The penalty amount which the seller owes is assumed to be proportional to the usual option payoff. We proved some propositions for the form of the early exercise regions. Using these propositions along with the flatness of the exercise boundaries we found the optimal strategies for both participants. After that we derived the closed form formulas for the fair option prices. The obtained results are illustrated by some numerical examples.

As we mentioned above, the results for the form of the exercise regions, derived in Section 3, are valid when the maturity is finite too. The main and very important difference is that we cannot expect that the optimal boundaries are flat. This is due to the existing forced exercise at the maturity. Our expectation for the form of the buyer's exercise boundary is to be similar to the exercise boundary of an American option. Also, we expect that the seller's exercise boundary is the strike near the maturity and changes its behavior when the time to maturity goes above some level. When the maturity is large the boundaries values have to tend to the values derived in this paper. The problem for deriving the exact exercise boundaries and the fair option prices is open. We cannot expect the existence of closed form formulas.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Tsvetelin S. Zaevski: Conceptualization, Methodology, Formal analysis, Software, Investigation, Writing - original draft, Writing - review & editing.

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Appendix A. Some propositions

Proposition A.1. *Let τ be the first hitting time of a Brownian motion with drift μ to the positive level a. Then the Laplace transform of its distribution is*

1. *If $a > 0$ then*
$$E\left[e^{-y\tau} I_{\tau<\infty}\right] = e^{-\left(\sqrt{\mu^2+2y}-\mu\right)a}. \tag{A.1}$$
2. *If $a < 0$ then*
$$E\left[e^{-y\tau} I_{\tau<\infty}\right] = e^{\left(\sqrt{\mu^2+2y}+\mu\right)a}. \tag{A.2}$$

Proposition A.2. *The Laplace transforms of the first exit time of a Brownian motion with drift μ from a strip (a, b) are presented by*

$$E^x\left[e^{-y\tau} I_{\tau=a}\right] = e^{\mu(a-x)} \frac{\sinh\left((b-x)\sqrt{2y+\mu^2}\right)}{\sinh\left((b-a)\sqrt{2y+\mu^2}\right)} \tag{A.3}$$

$$E^x\left[e^{-y\tau} I_{\tau=b}\right] = e^{\mu(b-x)} \frac{\sinh\left((x-a)\sqrt{2y+\mu^2}\right)}{\sinh\left((b-a)\sqrt{2y+\mu^2}\right)}. \tag{A.4}$$

Appendix B. Uniqueness of the solutions

B1. Call case: seller's boundary

We shall prove that derivative (16) has only one root in the interval (0, 1). Let the functions $m(a)$, $l(a)$, and $h(a;\ \eta)$ be defined as

$$m(a) = a^{p+1}(p-q-1) - a^p k(p-q) + a(q+1) - qk \tag{B.1}$$

$$l(a) = -a^{p-q}p(1-k) \tag{B.2}$$

$$h(a;\eta) = \eta m(a) + l(a). \tag{B.3}$$

We have to show that function (B.3) has just one root. We shall examine first the udiscounted case $p = q + 1$. We shall omit for simplicity the mark η. Function (B.3) turns to

$$h(a) = -a^p\eta k + ap(\eta - 1 + k) - (p-1)k\eta. \tag{B.4}$$

Its derivative is

$$h_a(a) = -a^{p-1}p\eta k + p(\eta - 1 + k). \tag{B.5}$$

Obviously, it is decreasing and since $h_a(1) = p(1-k)(\eta - 1) > 0$, the derivative $h_a(a)$ is positive in the whole interval (0, 1). Therefore the function $h(a)$ is increasing. Since $h(0) = -(p-1)k\eta < 0$ and $h(1) = p(1-k)(\eta-1) > 0$, the function $h(a)$ has just one root.

Now suppose that $p > q + 1$. We shall use the following two lemmas.

Lemma B.1. *When the function $h(a;\ 1)$ increases, the functions $m(a)$ and $h(a;\ \eta)$ increase too.*

Proof. Suppose that the function $h(a;\ 1)$ increases in some point a. Since the function $l(a)$ is decreasing, the function $m(a)$ has to be increasing. Thus the function $h(a;\eta) = (\eta - 1)m(a) + h(a;1)$ is a sum of two increasing functions and therefore it is increasing too. □

Lemma B.2. *If $h(\bar{a};1) = h(\bar{a};\eta)$ for some $\bar{a}$, then $h(\bar{a};\eta) = h(\bar{a};1) < 0$.*

Proof. If $h(\bar{a};1) = h(\bar{a};\eta)$, then $m(\bar{a}) = 0$. Hence, $h(a;1) = h(\bar{a};\eta) = l(\bar{a}) < 0$. □

We shall examine now the behavior of the function $h(a;\ 1)$. Its derivatives are

$$\begin{aligned} h_a(a;1) &= a^p(p+1)(p-q-1) - a^{p-1}pk(p-q) \\ &\quad - a^{p-q-1}(p-q)p(1-k) + (q+1) \\ h_{aa}(a;1) &= a^{p-q-2}p\begin{bmatrix} a^{q+1}(p+1)(p-q-1) - a^q(p-1)k(p-q) \\ -(p-q-1)(p-q)(1-k) \end{bmatrix}. \end{aligned} \tag{B.6}$$

Let us define the function $n(\ \cdot\ )$ as

$$\begin{aligned} n(a) &= a^{q+1}(p+1)(p-q-1) - a^q(p-1)k(p-q) \\ &\quad - (p-q-1)(p-q)(1-k). \end{aligned} \tag{B.7}$$

Its derivative is

$$n_a(a) = a^{q-1}[a(q+1)(p+1)(p-q-1) - q(p-1)k(p-q)] \tag{B.8}$$

and its positive root is

$$\bar{a} = \frac{q(p-1)k(p-q)}{(q+1)(p+1)(p-q-1)}. \tag{B.9}$$

Suppose first that $\bar{a} \geq 1$. We shall call this the case (A). Then the derivative $n_a(a)$ is negative in the whole interval (0, 1) and therefore the function $n(a;\ 1)$ is decreasing. Thus the derivative $h_{aa}(a;\ 1)$ is negative since $n(0) < 0$. Hence, the function $h_a(a;\ 1)$ is decreasing and therefore it is positive since $h_a(1;1) = 0$. Thus we conclude that the function $h(a;\ 1)$ increases from $-qk$ to 0 in the interval (0, 1).

Now suppose that $\bar{a} < 1$ which we shall call the case (B). Thus the derivative $n_a(a)$ is negative in the interval $(0, \bar{a})$ and positive in the interval $(\bar{a}, 1)$. Therefore the function $n(a)$ starts from a negative value, decreases to a minimum and then increases. If $n(1) \leq 0$, then we are in the case (A). Suppose that $n(1) > 0$. Hence, the derivative $h_a(a;\ 1)$ starts from the positive value $h_a(0;1) = q + 1$, decreases to a negative minimum, and then increases to zero staying negative. Thus we conclude that it is first positive and then negative. Therefore the function $h(a;\ 1)$ start from the negative value $-qk$, increases to a positive maximum having a zero and after that decreases to 0.

If we are in the case (A), Lemma B.1 gives us that the function $h(a;\ \eta)$ increases too. Since $h(0) = -qk\eta < 0$ and $h(1;\ \eta) = p(1-k)(\eta-1) > 0$, it has just one zero in the interval (0, 1).

Suppose now that we are in the case (B). Let us denote by a^* the maximum of the function $h(a;\ 1)$. Since $h(a;\ 1)$ increases in the interval $(0,\ a^*)$, Lemma B.1 gives us that the function $h(a;\ \eta)$ increases too. Therefore it can have only one root in the interval $(0,\ a^*)$. Lemma B.2 shows that the function $h(a;\ \eta)$ has no roots in the interval $(a^*,\ 1)$ since $h(a;\ 1)$ is positive and $h(1;\eta) = p(1-k)(\eta-1) > h(1;1) = 0$. Therefore the function $h(a;\ \eta)$ has just one root.

B2. Call case: buyer's boundary

We shall prove that derivative (19) has just one root larger than one except if (A) $\lambda = 0$ or (B) $\lambda > 0$, $k = 1$, and $r < 0$. Let the function $h(\cdot)$ be defined as

$$h(b) = -b^{p+1}(p-q-1) + b^p k(p-q) + b^{p-q}p(1-k)\eta - b(q+1) + qk. \tag{B.10}$$

Suppose first that $\lambda = 0$ or equivalently $p = q+1$. Then function (B.10) turns to

$$h(b) = b^p k + bp[(1-k)\eta - 1] + (p-1)k \tag{B.11}$$

and its derivative is

$$h_b(b) = p\left[b^{p-1}k + (1-k)\eta - 1\right]. \tag{B.12}$$

It is positive in the whole interval $(1, \infty)$, since $h_b(1) = p(1-k)(\eta-1) \geq 0$ and therefore function (B.11) is increasing. Since $h(1) = p(1-k)(\eta-1) \geq 0$, the function $h(b)$ is positive when b is larger than one.

Now suppose that $p > q+1$. The derivatives of function (B.10) are

$$\begin{aligned} h_b(b) &= -b^p(p+1)(p-q-1) + b^{p-1}pk(p-q) \\ &\quad + b^{p-q-1}(p-q)p(1-k)\eta - (q+1) \\ h_{bb}(b) &= b^{p-q-2}p\begin{bmatrix} -b^{q+1}(p+1)(p-q-1) + b^q(p-1)k(p-q) \\ +(p-q-1)(p-q)(1-k)\eta \end{bmatrix}. \end{aligned} \tag{B.13}$$

Suppose first, that $k = 1$. Then the second derivative (B.13) turns to

$$h_{bb}(b) = b^{p-2}p[-b(p+1)(p-q-1) + (p-1)(p-q)] \tag{B.14}$$

and it has a root

$$\bar{b} = \frac{(p-1)(p-q)}{(p+1)(p-q-1)}. \tag{B.15}$$

If we suppose that the risk free rate is negative, Lemma 4.1 gives us that $2q+1 < p$. It turns out that this is equivalent to $\bar{b} < 1$. Therefore $h_{bb}(b)$ is negative in the whole interval $(1,\ \infty)$. So, $h_b(b)$ is a decreasing function and since $h_b(1) = 0$, it is negative. Analogously, we can conclude that $h(b)$ is negative too, since $h(1) = 0$.

Otherwise, if $r > 0$, then $\bar{b} > 1$. Hence, $h_{bb}(b)$ is positive in the interval $\left(1, \bar{b}\right)$ and negative in the interval $\left(\bar{b}, \infty\right)$. Therefore, the function $h_b(b)$ starts from zero, increases to a positive maximum and then decreases to minus infinity. Thus it has a root, before which it is positive and after which it is negative. Analogously, we can conclude that $h(b)$ has just one root larger than one.

Suppose now that $k < 1$. Let the function $l(\cdot)$ be defined as

$$l(b) = -b^{q+1}(p+1)(p-q-1) + b^q(p-1)k(p-q) + (p-q-1)(p-q)(1-k)\eta. \tag{B.16}$$

Its derivative is

$$l_b(b) = b^{q-1}[-b(q+1)(p+1)(p-q-1) + q(p-1)k(p-q)] \tag{B.17}$$

and it has a root

$$\bar{b} = \frac{q(p-1)k(p-q)}{(q+1)(p+1)(p-q-1)}. \tag{B.18}$$

Suppose first that $\bar{b} \leq 1$. Then $l_b(b) \leq 0$ for $b \in (1,\ \infty)$. Therefore the function $l(b)$ is decreasing and thus it can has at most one root larger than one. Suppose that it has a root. Then $l(b)$ is positive before it and negative after that. Hence, the derivative $h_b(b)$ starts from the positive value $h_b(1) = p(1-k)(p-q)(\eta-1)$, increases to a maximum and after that decreases to minus infinity. Otherwise, if the function $l(b)$ has not a root, then it is negative in the whole interval $(1,\ \infty)$. Thus $h_b(b)$ starts from the positive value and then decreases to minus infinity. In both cases it is first positive and after that negative. The same arguments show that the function $h(b)$ has the same behavior and thus it has just one root in the interval $(1,\ \infty)$.

Suppose now that $\bar{b} > 1$. Then $l_b(b) > 0$ for $b \in \left(0, \bar{b}\right)$ and $l_b(b) < 0$ for $b \in \left(\bar{b}, \infty\right)$. Therefore the function $l(b)$ starts from a positive value $l(1) > l(0) = (p-q-1)(p-q)(1-k)\eta$, increases to a maximum and after that decreases to minus infinity. This behavior we had already studied above.

B3. Put case: buyer's boundary

We shall prove that derivative (36) has a unique root in the interval (0, 1) except (A) when $\lambda = 0$ and $k = 1$ or (B) when $\lambda > 0$, $k = 1$, and $r > 0$. Let the function $h(\,\cdot\,;\ \cdot\,)$ be defined as

$$h(a;\eta) = -a^{p+1}(p-q-1) + a^p k(p-q) - a^{p-q}p(k-1)\eta - a(q+1) + qk. \tag{B.19}$$

We shall examine first the undiscounted case $\lambda = 0$ or equivalently $p = q + 1$. We shall omit the notation η for simplicity. Function (B.19) turns to

$$h(a) = a^p k - ap[(k-1)\eta + 1] + qk. \tag{B.20}$$

Its derivative is

$$h_a(a) = p\Big(a^{p-1}k - (k-1)\eta - 1\Big). \tag{B.21}$$

We have that the derivative $h_a(a)$ is negative in the interval (0, 1) since $h_a(1) = -p(k-1)(\eta-1) \le 0$. Therefore the function $h(a)$ is decreasing. We have $h(0) = qk > 0$ and $h(1) = -p(k-1)(\eta-1) \le 0$. Hence, the function $h(a)$ has just one root in the interval (0, 1) except in the case $k = 1$, in which it is positive.

Now suppose that $p > q + 1$. The derivatives of function (B.19) are

$$\begin{aligned} h_a(a;\eta) &= -a^p(p+1)(p-q-1) + a^{p-1}pk(p-q) \\ &\quad - a^{p-q-1}(p-q)p(k-1)\eta - (q+1) \\ h_{aa}(a;\eta) &= a^{p-q-2}p\begin{bmatrix} -a^{q+1}(p+1)(p-q-1) + a^q(p-1)k(p-q) \\ -(p-q-1)(p-q)(k-1)\eta \end{bmatrix}. \end{aligned} \tag{B.22}$$

Suppose that $k = 1$. Then the second derivative turns to

$$h_{aa}(a;\eta) = a^{p-2}p[-a(p+1)(p-q-1) + (p-1)(p-q)] \tag{B.23}$$

and it is zero for

$$\bar{a} = \frac{(p-1)(p-q)}{(p+1)(p-q-1)}. \tag{B.24}$$

If the risk free rate is positive, then Lemma 4.1 gives us that $2q + 1 > p$ which is equivalent to $\bar{a} > 1$. Hence, the derivative $h_{aa}(a;\ \eta)$ is positive in the interval (0, 1) and therefore the first derivative $h_a(a;\ \eta)$ is an increasing function. Since $h_a(1;\eta) = 0$, the derivative $h_a(a;\ \eta)$ is negative and thus the function $h(a;\ \eta)$ is decreasing. Since $h(1;\eta) = 0$ too, the function $h(a;\ \eta)$ is positive.

Otherwise, if the risk free rate is negative, then $\bar{a} < 1$. Therefore the second derivative $h_{aa}(a;\ \eta)$ is positive in the interval $(0, \bar{a})$ and negative in the interval $(\bar{a}, 1)$. Hence, the first derivative $h_a(a;\ \eta)$ starts from the negative value $h_a(0;\eta) = -(q+1)$, increases to a maximum and then decreases to zero. Thus it has just one root before which it is negative and it is positive after that. Therefore the function $h(a;\ \eta)$ starts from the positive value $h(0;\eta) = q$, decreases to a minimum and then increases to zero. Therefore it has just one root in the interval (0, 1).

Suppose now that $k > 1$. First we shall examine the case $\eta = 1$. Note that $h_a(a;\ \eta) < h_a(a;\ 1)$. Let us define the function $l(\,\cdot\,;\ \cdot\,)$ as

$$\begin{aligned} l(a) &= -a^{q+1}(p+1)(p-q-1) + a^q(p-1)k(p-q) \\ &\quad - (p-q-1)(p-q)(k-1). \end{aligned} \tag{B.25}$$

Its derivative is

$$l_a(a) = a^{q-1}[-a(q+1)(p+1)(p-q-1) + q(p-1)k(p-q)]. \tag{B.26}$$

The positive root of this derivative is

$$\bar{a} = \frac{q(p-1)k(p-q)}{(q+1)(p+1)(p-q-1)}. \tag{B.27}$$

Suppose that $\bar{a} \ge 1$. Then $l_a(a) > 0$ for all $a \in (0, 1)$. Therefore the function $l(a)$ is increasing which means that it has at most one root less than one. If it has not a root, then $l(a) < 0$ for all $a \in (0, 1)$, since $l(0) < 0$. Thus the derivative $h_a(a;\ 1)$ is decreasing.[3] Since $h_a(0;1) = -(q+1) < 0$, the derivative $h_a(a;\ 1)$ is negative in the whole interval (0, 1). Otherwise, if the function $l(a)$ has one root, it has to be first negative and after that positive. Thus the derivative $h_a(a;\ 1)$ starts from the negative value $h_a(0;1) = -(q+1)$, decreases to a minimum and then increases to zero staying negative. Thus we can conclude that in both cases the derivative $h_a(a;\ 1)$ is negative. Since $h_a(a;\ \eta) < h_a(a;\ 1)$, the derivative $h_a(a;\ \eta)$ is negative too and therefore the function $h(a;\ \eta)$ is decreasing. Since $h(0;\eta) = qk > 0$ and $h(1;\eta) = -p(k-1)(\eta-1) < 0$, the function $h(a;\ \eta)$ has just one root in the interval (0, 1).

[3] This case is possible only when $\eta > 0$.

Suppose now that $\bar{a} < 1$. Hence, the derivative $l_a(a)$ is positive in the interval $(0, \bar{a})$ and negative in the interval $(\bar{a}, 1)$. So, the function $l(a)$ starts from the negative value, increases to a maximum and after that decreases. Let us denote by a^* this maximum. If $l(a^*) \leq 0$ or $l(1) > 0$ we are in the previous case. Suppose that $l(a^*) > 0$ and $l(1) < 0$. Therefore the derivative $h_a(a; 1)$ starts from the negative value $h_a(0; 1) = -(q+1)$, decreases to a negative minimum, increases to a positive maximum and then decreases to zero. Hence, it is first negative and after that positive. Thus the function $h(a; 1)$ starts from the positive value $h(0; 1) = qk$, decreases to a negative minimum and after that increases to zero. Note that $h_a(a; \eta) < h_a(a; 1)$. This means that when $h(a; 1)$ decreases, $h(a; \eta)$ decreases faster. Also if $h(a; 1)$ increases, then $h(a; \eta)$ increases slower or decreases. Thus we conclude that the function $h(a; \eta)$ has just one root in the interval (0, 1) even tough that it may be not monotone.

B4. Put case: seller's boundary

We shall prove now that derivative (34) has only one root in the interval $(1, \infty)$ except in the undiscounted case. Let the function $h(b; \eta)$ be defined as

$$\begin{aligned} h(b; \eta) &= b^{p+1}\eta(p-q-1) - b^p\eta k(p-q) \\ &\quad + b^{p-q}p(k-1) + b(q+1)\eta - q\eta k. \end{aligned} \tag{B.28}$$

Suppose first that $\lambda = 0$ or equivalently $p = q+1$. We shall omit for simplicity the mark η. Thus function (B.28) turns to

$$h(b) = -b^p\eta k + bp(k-1+\eta) - (p-1)\eta k. \tag{B.29}$$

Its derivative is

$$h_b(b) = -b^{p-1}p\eta k + p(k-1+\eta). \tag{B.30}$$

Since $h_b(1) = -p(k-1)(\eta-1) < 0$, we have that function (B.29) is decreasing in the interval $(1, \infty)$. Since $h(1) = -p(k-1)(\eta-1) < 0$ too, the function $h(b)$ is negative in the interval $(1, \infty)$.

Now suppose that $p > q+1$. We change the variable as $d = \frac{1}{b}$. Thus the function (B.28) turns to

$$h(d; \eta) = \frac{1}{d^{p+1}}\begin{bmatrix} -d^{p+1}q\eta k + d^p(q+1)\eta + d^{q+1}p(k-1) \\ -d\eta k(p-q) + \eta(p-q-1) \end{bmatrix}. \tag{B.31}$$

Let the function $\bar{h}(d; \eta)$ be defined as

$$\begin{aligned} \bar{h}(d; \eta) &= -d^{p+1}q\eta k + d^p(q+1)\eta + d^{q+1}p(k-1) \\ &\quad - d\eta k(p-q) + \eta(p-q-1). \end{aligned} \tag{B.32}$$

Let us define the functions $m(d)$ and $l(d)$ as

$$m(d) = -d^{p+1}qk + d^p(q+1) - dk(p-q) + (p-q-1) \tag{B.33}$$

$$l(d) = d^{q+1}p(k-1). \tag{B.34}$$

Thus the function $\bar{h}(d; \eta)$ turns to

$$\bar{h}(d; \eta) = \eta m(d) + l(d). \tag{B.35}$$

We shall prove the following two lemmas.

Lemma B.3. *When the function $\bar{h}(d; 1)$ decreases, the functions $m(d)$ and $\bar{h}(d; \eta)$ decrease too.*

Proof. Suppose that the function $\bar{h}(d; 1)$ decreases in some point d. Since the function $l(d)$ is increasing, the function $m(d)$ has to be decreasing. Thus the function $\bar{h}(d; \eta) = (\eta-1)m(d) + \bar{h}(d; 1)$ is a sum of two decreasing functions and therefore is decreasing too. □

Lemma B.4. *If $\bar{h}(\bar{d}; 1) = \bar{h}(\bar{d}; \eta)$ for some $\bar{d}$, then $\bar{h}(\bar{d}; \eta) = \bar{h}(\bar{d}; 1) > 0$.*

Proof. If $\bar{h}(\bar{d}; 1) = \bar{h}(\bar{d}; \eta)$, then $m(\bar{d}) = 0$. Hence, $\bar{h}(\bar{d}; 1) = \bar{h}(\bar{d}; \eta) = l(\bar{d}) > 0$. □

The derivatives of function $\bar{h}(d; 1)$ are

$$\begin{aligned} \bar{h}_d(d; 1) &= -d^p(p+1)qk + d^{p-1}p(q+1) \\ &\quad + d^q(q+1)p(k-1) - k(p-q) \\ \bar{h}_{dd}(d; 1) &= d^{q-1}p\begin{bmatrix} -d^{p-q}(p+1)qk + d^{p-q-1}(p-1)(q+1) \\ +q(q+1)(k-1) \end{bmatrix}. \end{aligned} \tag{B.36}$$

Let the function $n(\cdot)$ be defined as

$$n(d) = -d^{p-q}(p+1)qk + d^{p-q-1}(p-1)(q+1) + q(q+1)(k-1). \tag{B.37}$$

Its derivative is

$$n_d(d) = d^{p-q-2}[-d(p-q)(p+1)qk + (p-q-1)(p-1)(q+1)]. \tag{B.38}$$

It has a positive root

$$\bar{d} = \frac{(p-q-1)(p-1)(q+1)}{(p-q)(p+1)qk}. \tag{B.39}$$

Suppose first, that $\bar{d} \geq 1$. This means that the derivative $n_d(d)$ is positive in the whole interval (0, 1) and therefore the function $\bar{h}_{dd}(d;1)$ is increasing. Since $\bar{h}_{dd}(0^+;1) > 0$, the first derivative $\bar{h}_d(d;1)$ is an increasing function too. We have that $\bar{h}_d(1;1) = 0$ and therefore the function $\bar{h}(d;1)$ is decreasing. Hence in the first case (A) it decreases from $(p-q-1)$ to 0.

Suppose now, that $\bar{d} < 1$. Hence the derivative $n_d(d)$ is positive in the interval $\left(0, \bar{d}\right)$ and negative in the interval $\left(\bar{d}, 1\right)$. Hence the function $\bar{h}_{dd}(d;1)$ starts from zero, increases to a maximum and then decreases. If $\bar{h}_{dd}(1;1) \geq 0$, then we are in the case (A). Suppose that $\bar{h}_{dd}(1;1) < 0$. Then the first derivative $\bar{h}_d(d;1)$ starts from the negative value $-k(p-q)$ increases to a positive maximum and then decreases to zero. Thus we can conclude that the function $\bar{h}(d;1)$ starts from the positive value $(p-q-1)$, decreases to a negative minimum and after that increases to zero. We shall denote that case by (B)

If we are in the case (A), Lemma B.3 gives us that the function $\bar{h}(a;\eta)$ decreases too. Since $\bar{h}(0;\eta) = \eta(p-q-1) > 0$ and $h(1;\eta) = -p(k-1)(\eta-1) < 0$, the function has just one zero.

Suppose now that we are in the case (B). Let us denote by d^* the minimum of the function $\bar{h}(d;1)$. Since $h(d;1)$ decreases in the interval $(0, d^*)$, Lemma B.3 gives us that the function $\bar{h}(d;\eta)$ decreases too. Therefore it can have only one root in the interval $(0, d^*)$. Lemma B.4 shows that the function $\bar{h}(d;\eta)$ has no roots in the interval $(d^*, 1)$ since $\bar{h}(d;1)$ is negative and $\bar{h}(1;\eta) = -p(k-1)(\eta-1) < \bar{h}(1;1) = 0$. Therefore the function $\bar{h}(d;\eta)$ has just one root in the interval (0, 1) and thus function (B.28) has a unique root in the interval $(1, \infty)$.

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